

STEMS AND SPECTRAL SEQUENCES

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ABSTRACT. We introduce the category $\mathcal{P}stem[n]$ of n -stems, with a functor $\mathcal{P}[n]$ from spaces to $\mathcal{P}stem[n]$. This can be thought of as the n -th order homotopy groups of a space. We show how to associate to each simplicial n -stem Q_\bullet an $(n+1)$ -truncated spectral sequence. Moreover, if $Q_\bullet = \mathcal{P}[n]X_\bullet$ is the Postnikov n -stem of a simplicial space X_\bullet , the truncated spectral sequence for Q_\bullet is the truncation of the usual homotopy spectral sequence of X_\bullet . Similar results are also proven for cosimplicial n -stems. They are helpful for computations, since n -stems in low degrees have good algebraic models.

0. INTRODUCTION

Many of the spectral sequences of algebraic topology arise as the homotopy spectral sequence of a (co)simplicial space – including the spectral sequence of a double complex, the (stable or unstable) Adams spectral sequence, the Eilenberg-Moore spectral sequence, and so on (see §4.14). Given a simplicial space X_\bullet , the E^2 -term of its homotopy spectral sequence has the form $E_{s,t}^2 = \pi_s \pi_t X_\bullet$, so it may be computed by applying the homotopy group functor dimensionwise to X_\bullet .

In this paper we show that the higher terms of this spectral sequence are obtained analogously by applying ‘higher homotopy group’ functors to X_\bullet . These functors are given explicitly in the form of certain *Postnikov stems*, defined in Section 1; the Postnikov 0-stem of a space is equivalent to its homotopy groups.

We then show how the E^r -term of the homotopy spectral sequence of a simplicial space X_\bullet can be described in terms of the $(r-2)$ -Postnikov stem of X_\bullet , for each $r \geq 2$ (see Theorem 3.14) – and similarly for the homotopy spectral sequence of a cosimplicial space X^\bullet (see Theorem 4.12).

As an application for the present paper, in [BB2] we generalize the first author’s result with Mamuka Jibladze (in [BJ]), which shows that the E^3 -term of the stable Adams spectral sequence can be identified as a certain secondary derived functor Ext . We do this by showing how to define in general the *higher order derived functors* of a continuous functor $F : \mathcal{C} \rightarrow \mathcal{T}_*$, by applying F to a simplicial resolution W_\bullet in \mathcal{C} , and taking Postnikov n -stems of FW_\bullet .

0.1. Notation and conventions. The category of pointed connected topological spaces will be denoted by \mathcal{T}_* ; that of pointed sets by Set_* ; that of groups by $\mathcal{G}p$.

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For any category \mathcal{C} , $s\mathcal{C}$ denotes the category of simplicial objects over \mathcal{C} , and $c\mathcal{C}$ that of cosimplicial objects over \mathcal{C} . However, we abbreviate $sSet$ to \mathcal{S} , $sSet_*$ to \mathcal{S}_* , and $s\mathcal{G}p$ to \mathcal{G} . The constant (co)simplicial object on an object $X \in \mathcal{C}$ is written $c(X)_\bullet \in s\mathcal{C}$ (respectively, $c(X)^\bullet \in c\mathcal{C}$). For any small indexing category I , the category of functors $I \rightarrow \mathcal{C}$ is denoted by \mathcal{C}^I .

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1. POSTNIKOV STEMS

The Postnikov system of a topological space (or simplicial set) X is the tower of fibrations:

$$(1.1) \quad \dots \rightarrow P^{n+1}X \xrightarrow{p^{n+1}} P^nX \xrightarrow{p^n} P^{n-1}X \dots P^1X \xrightarrow{p^1} P^0X,$$

equipped with maps $q^n : X \rightarrow P^nX$ (with $p^n \circ q^n = q^{n-1}$), which induce isomorphisms on homotopy groups in degrees $\leq n$. Here P^nX is n -coconnected (that is, $\pi_i P^nX = 0$ for $i > n$) and $\pi_i p^n$ is an isomorphism for $i < n$. The fiber of the map $p^n : P^nX \rightarrow P^{n-1}X$ is the Eilenberg-Mac Lane space $K(\pi_n X, n)$, so the fibers are determined up to homotopy by $\pi_* X$. Thus a generalization of the homotopy groups of X is provided by the following notion:

1.2. **Definition.** For any $n \geq 0$, a *Postnikov n -stem* in \mathcal{T}_* is a tower:

$$(1.3) \quad \mathcal{Q} := \left(\dots \rightarrow Q_{k+1} \xrightarrow{q_{k+1}} Q_k \xrightarrow{q_k} Q_{k-1} \dots Q_0 \right)$$

in $\mathcal{T}_*^{(\mathbb{N}, \leq)}$, in which Q_k is $(k-1)$ -connected and $(n+k)$ -coconnected (so that $\pi_i(Q_k) = 0$ for $i < k$ or $i > n+k$) and $\pi_i(q_k)$ is an isomorphism for $k \leq i < n+k$. Here (\mathbb{N}, \leq) is the usual linearly ordered category of the natural numbers. The space Q_k is called the k -th n -window of \mathcal{Q} .

Such an n -stem is thus a collection of overlapping $(k-1)$ -connected $n+k$ -types, which may be depicted for $n=2$ as follows:

$$\begin{array}{ccccccc} \dots & * & * & * & & & \\ & & * & * & * & & \\ & & & * & * & * & \\ & & & & * & * & * \dots \end{array}$$

where each row exhibits the $n+1$ non-trivial homotopy groups (denoted by $*$) of one n -window, and all those in the i -th column (corresponding to π_i) are isomorphic.

We denote by $\mathcal{P}stem[n]$ the full subcategory of Postnikov n -stems in the functor category $\mathcal{T}_*^{(\mathbb{N}, \leq)}$ (with model category structure on the latter as in [Hi, 11.6]). Thus the morphisms in $\mathcal{P}stem[n]$ are given by strictly commuting maps of towers, and $f : \mathcal{Q} \rightarrow \mathcal{Q}'$ is a weak equivalence (respectively, a fibration) if each $f_k : Q_k \rightarrow Q'_k$ is such. This lets us define the homotopy category of Postnikov n -stems, $ho \mathcal{P}stem[n]$, as a full sub-category of $ho \mathcal{T}_*^{(\mathbb{N}, \leq)}$.

The category $\mathcal{P}stem[n]$ is pointed, has products, and is equipped with canonical functors

$$(1.4) \quad \begin{array}{c} \mathcal{T}_* \\ \swarrow \mathcal{P}[n] \quad \searrow \mathcal{P}[n-1] \quad \searrow \mathcal{P}[0] \\ \dots \mathcal{Pstem}[n] \xrightarrow{\overline{\mathcal{P}}[n-1]} \mathcal{Pstem}[n-1] \xrightarrow{\overline{\mathcal{P}}[n-2]} \dots \xrightarrow{\overline{\mathcal{P}}[0]} \mathcal{Pstem}[0] \end{array}$$

which preserve products and weak equivalences.

1.5. *Remark.* The sequence of functors (1.4) is described by a commuting diagram, in which we may take all maps to be fibrations:

$$(1.6) \quad \begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \nearrow & & \nearrow & & \nearrow \\ \dots & \longrightarrow & Q_{k+1}^{n+k+1} & \xrightarrow{q_{k+1}^n} & Q_k^{n+k} & \xrightarrow{q_k^n} & Q_{k-1}^{n+k-1} \longrightarrow \dots \longrightarrow Q_0^n \\ & & \downarrow p_{k+1}^n & & \downarrow p_k^n & & \downarrow p_{k-1}^n \\ \dots & \longrightarrow & Q_{k+1}^{n+k} & \xrightarrow{q_{k+1}^{n-1}} & Q_k^{n+k-1} & \xrightarrow{q_k^{n-1}} & Q_{k-1}^{n+k-2} \longrightarrow \dots \longrightarrow Q_0^{n-1} \\ & & \downarrow p_{k+1}^{n-1} & & \downarrow p_k^{n-1} & & \downarrow p_{k-1}^{n-1} \\ \dots & \longrightarrow & Q_{k+1}^{n+k-1} & \xrightarrow{q_{k+1}^{n-2}} & Q_k^{n+k-2} & \xrightarrow{q_k^{n-2}} & Q_{k-1}^{n+k-3} \longrightarrow \dots \longrightarrow Q_0^{n-2} \\ & & \vdots & & \vdots & & \vdots \\ \dots & \longrightarrow & Q_{k+1}^{k+1} & \xrightarrow{q_{k+1}^0} & Q_k^k & \xrightarrow{q_k^0} & Q_{k-1}^{k-1} \longrightarrow \dots \longrightarrow Q_0^0 \end{array}$$

Here $\pi_i Q_k^n = 0$ for $i < k$ or $i > n$, and all maps induce isomorphisms in π_i whenever possible. Thus:

- (a) The k -th column (from the right) is the Postnikov tower for $Q_k := \lim_n Q_k^n$.
- (b) The diagonals are the dual Postnikov system of connected covers for Q_0^j .
- (c) The n -th row (from the bottom) is a Postnikov n -stem.
- (d) In particular, each space in the 0-stem (the bottom row) is an Eilenberg-Mac Lane space, and the maps q_k^0 are nullhomotopic. Thus the homotopy type of the bottom line in $\text{ho } \mathcal{Pstem}[0]$ is determined by the collection of homotopy groups $\{\pi_k Q_k^k\}_{k=0}^\infty$.

1.7. **Definition.** The motivating example of a Postnikov n -stem is a *realizable* one, associated to a space $X \in \mathcal{T}_*$, and denoted by $\mathcal{P}[n]X$, with $(\mathcal{P}[n]X)_k := P^{n+k}X\langle k \rangle$. As usual, $Y\langle k \rangle$ denotes the $(k-1)$ -connected cover of a space $Y \in \mathcal{T}_*$. Each fibration $q_k : (\mathcal{P}[n]X)_k \rightarrow (\mathcal{P}[n]X)_{k-1}$ fits into a commuting triangle of fibrations:

$$(1.8) \quad \begin{array}{ccc} P^{n+k+1}X\langle k+1 \rangle & \xrightarrow{q} & P^{n+k}X\langle k \rangle \\ & \searrow p \quad \nearrow r & \\ & P^{n+k}X\langle k+1 \rangle & \end{array}$$

in which the maps p and r are the fibration of (1.1) and the covering map, respectively. See [BB1, §10.5] for a natural context in which non-realizable Postnikov n -stems arise.

1.9. Examples of stems. The functor $\mathcal{P}[0]_* : \mathcal{T}_* \rightarrow \text{ho } \mathcal{Pstem}[0]$ induced by $\mathcal{P}[0]$ is equivalent to the homotopy group functor: in fact, the homotopy groups of a space define a functor $\pi_* : \mathcal{T}_* \rightarrow \mathcal{K}$ into the product category $\mathcal{K} := \prod_{i=0}^{\infty} \mathcal{K}_i$, where $\mathcal{K}_0 = \text{Set}_*$, $\mathcal{K}_1 = \mathcal{G}p$, and $\mathcal{K}_i = \text{Ab}\mathcal{G}p$, for $i \geq 2$. Moreover, there is an equivalence of categories $\vartheta : \mathcal{K} \cong \text{ho } \mathcal{Pstem}[0]$, such that the functor $\mathcal{P}[0]_*$ is equivalent to the composite functor $\vartheta \circ \pi_* : \mathcal{T}_* \rightarrow \mathcal{K}$.

Similarly, the functor $\mathcal{T}_* \rightarrow \text{ho } \mathcal{Pstem}[1]$ induced by $\mathcal{P}[1]$ is equivalent to the secondary homotopy group functor of [BM, §4], in the sense that each secondary homotopy group $\pi_{n,*}X$ completely determines the n -th 1-window of X . Using the results on secondary homotopy groups in [BM], one obtains a homotopy category of algebraic 1-stems which is equivalent to $\text{ho } \mathcal{Pstem}[1]$.

A category of algebraic models for 2-stems is only partially known. The homotopy classification of $(k-1)$ -connected $(k+2)$ -types is described for all k in [Ba]; this theory can be used to classify homotopy types of Postnikov 2-stems.

2. THE SPECTRAL SEQUENCE OF A SIMPLICIAL SPACE

We begin with the construction of the homotopy spectral sequence for a simplicial space (cf. [Q], [BF, Theorem B.5], and [BK1, X, §6]), using the version given by Dwyer, Kan, and Stover in [DKSt2, §8] (see also [Bou2, §2.5], [Bou1], and [DKSt1, §3.6]). For this purpose, we require some explicit constructions for the E^2 -model category of simplicial spaces.

2.1. Definition. Given a simplicial object $X_{\bullet} \in s\mathcal{C}$, over a complete pointed category \mathcal{C} , for each $n \geq 1$ define its n -cycles object to be

$$Z_n X_{\bullet} := \{x \in X_n \mid d_i x = * \text{ for } i = 0, \dots, n\}.$$

Similarly, the the n -chains object for X_{\bullet} is

$$C_n X_{\bullet} := \{x \in X_n \mid d_i x = * \text{ for } i = 1, \dots, n\}$$

Set $Z_0 X_{\bullet} := X_0$. We denote the map $d_0|_{C_n X_{\bullet}} : C_n X_{\bullet} \rightarrow Z_{n-1} X_{\bullet}$ by $\mathbf{d}_0^{X_n}$.

2.2. Notation. For any non-negatively graded object T_* , we write ΩT_* for the graded object with $(\Omega T_*)_j := T_{j+1}$ for all $j \geq 0$. The notation is motivated by the natural isomorphism of graded groups $\pi_* \Omega X \cong \Omega(\pi_* X)$ for $X \in \mathcal{T}_*$.

2.3. Definition. Now assume that \mathcal{C} is a pointed model category of spaces, such as \mathcal{T}_* or \mathcal{G} , and X_{\bullet} is a Reedy fibrant simplicial object over \mathcal{C} – that is, for each $n \geq 1$, the universal face map $\delta_n : X_n \rightarrow M_n X_{\bullet}$ into the n -th matching object of X_{\bullet} is a fibration (see [Hi, 15.3]). The map $\mathbf{d}_0 = \mathbf{d}_0^{X_n}$ then fits into a fibration sequence in \mathcal{C} :

$$(2.4) \quad \cdots \Omega Z_n X_{\bullet} \rightarrow Z_{n+1} X_{\bullet} \xrightarrow{j_{n+1}^{X_{\bullet}}} C_{n+1} X_{\bullet} \xrightarrow{\mathbf{d}_0^{X_{n+1}}} Z_n X_{\bullet}$$

(see [DKSt2, Prop. 5.7]).

For each $n \geq 0$, the n -th *natural homotopy group* of the simplicial space X_\bullet , denoted by $\pi_n^{\natural} X_\bullet = \pi_{n,*}^{\natural} X_\bullet$, the cokernel of the map $(\mathbf{d}_0^{X_{n+1}})_\#$ (induced on homotopy groups by $\mathbf{d}_0^{X_{n+1}}$). Note that the cokernel of a maps of groups or pointed sets is generally just a pointed set.

We thus have an exact sequence of graded groups:

$$(2.5) \quad \pi_* C_{n+1} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+1}})_\#} \pi_* Z_n X_\bullet \xrightarrow{\hat{\vartheta}_n} \pi_{n,*}^{\natural} X_\bullet \rightarrow 0.$$

Together the groups $(\pi_{n,k}^{\natural} X_\bullet)_{n,k=0}^\infty$ constitute the *bigraded homotopy groups* of [DKSt2, §5.1].

2.6. Construction of the spiral sequence. Applying the functor π_* to the fibration sequence (2.4) yields a long exact sequence, with connecting homomorphism $\partial_\# : \Omega \pi_* Z_n X_\bullet = \pi_* \Omega Z_n X_\bullet \rightarrow \pi_* Z_{n+1} X_\bullet$. Note that the inclusion $\iota : C_n X_\bullet \hookrightarrow X_n$ induces an isomorphism $\iota_* : \pi_* C_n X_\bullet \cong C_n(\pi_* X_\bullet)$ for each $n \geq 0$ (see [Bl3, Prop. 2.7]). From (2.5) we see that:

$$\Omega \pi_n^{\natural} X_\bullet = \Omega \operatorname{Coker} (\mathbf{d}_0^{X_{n+1}})_\# \cong \operatorname{Im} \partial_\# \cong \operatorname{Ker} (j_{n+1}^{X_\bullet})_\# \subseteq \pi_* Z_{n+1} X_\bullet,$$

so we obtain a commutative diagram with exact rows and columns:

$$(2.7) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{Ker} (j_n)_* & \hookrightarrow & B_{n+1} X_\bullet & \xrightarrow{(j_n)_*} & B_{n+1} \pi_* X_\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega \pi_{n-1}^{\natural} X_\bullet & \xrightarrow{\ell_{n-1}} & \pi_* Z_n X_\bullet & \xrightarrow{(j_n^{X_\bullet})_\#} & Z_n \pi_* X_\bullet \longrightarrow \operatorname{Coker} h_n \longrightarrow 0 \\ & & \downarrow & \searrow s_n & \downarrow \hat{\vartheta}_n & & \downarrow \vartheta_n \\ 0 & \longrightarrow & \operatorname{Ker} h_n & \hookrightarrow & \pi_n^{\natural} X_\bullet & \xrightarrow{h_n} & \pi_n \pi_* X_\bullet \longrightarrow \operatorname{Coker} h_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

in which $B_{n+1} X_\bullet := \operatorname{Im} (\mathbf{d}_0^{X_{n+2}})_\# \subseteq \pi_* Z_n X_\bullet$ and $B_{n+1} \pi_* X_{n+2} := \operatorname{Im} \mathbf{d}_0^{\pi_* X_{n+2}}$ are the respective boundary objects. Note that the map $(j_n^{X_\bullet})_\# : \pi_* Z_n X_\bullet \rightarrow \pi_* C_n X_\bullet$ induced by the inclusion $j_n^{X_\bullet}$ of (2.4) above in fact factors through $Z_n \pi_* X_\bullet$, as indicated in the middle row of (2.7).

This defines the map of graded groups $h_n : \pi_n^{\natural} X_\bullet \rightarrow \pi_n(\pi_* X_\bullet)$. Note that for $n = 0$ the map $\hat{\iota}_*$ is an isomorphism, so h_0 is, too. The map $s_n : \Omega \pi_{n-1}^{\natural} X_\bullet \rightarrow \pi_n^{\natural} X_\bullet$ is the composite of the inclusion $\ell_{n-1} : \operatorname{Ker} (j_n^{X_\bullet})_\# \hookrightarrow \pi_* Z_n X_\bullet$ with the quotient map $\hat{\vartheta}_n : \pi_* Z_n X_\bullet \rightarrow \pi_n^{\natural} X_\bullet$ of (2.5), using the natural identification of $\Omega \pi_n^{\natural} X_\bullet$ with $\operatorname{Ker} (j_{n+1}^{X_\bullet})_\#$.

The map $\partial_{n+2} : \pi_{n+2} \pi_* X_\bullet \rightarrow \Omega \pi_n^{\natural} X_\bullet$ is induced by the composite

$$(2.8) \quad Z_{n+2} \pi_* X_\bullet \subseteq C_{n+2} \pi_* X_\bullet \cong \pi_* C_{n+2} X_\bullet \xrightarrow{(\mathbf{d}_0^{X_{n+2}})_\#} \pi_* Z_{n+1} X_\bullet,$$

which actually lands in $\text{Ker}(j_{n+1}^{X_\bullet})_\#$ by the exactness of the long exact sequence for the fibration (2.4).

These maps s_n , h_n , and ∂_n fit into a *spiral long exact sequence*:

$$(2.9) \quad \begin{aligned} \dots \rightarrow \Omega\pi_{n-1}^\natural X_\bullet &\xrightarrow{s_n} \pi_n^\natural X_\bullet \xrightarrow{h_n} \pi_n\pi_* X_\bullet \xrightarrow{\partial_n} \Omega\pi_{n-2}^\natural X_\bullet \\ &\xrightarrow{s_{n-1}} \pi_{n-1}^\natural X_\bullet \rightarrow \dots \rightarrow \pi_0^\natural X_\bullet \xrightarrow{\cong} \pi_0\pi_* X_\bullet. \end{aligned}$$

(cf. [DKSt2, 8.1]).

2.10. The spectral sequence of a simplicial space. For any simplicial space $X_\bullet \in s\mathcal{T}_*$ (or bisimplicial set), Bousfield and Friedlander showed that there is a first-quadrant spectral sequence of the form

$$(2.11) \quad E_{s,t}^2 = \pi_s\pi_t X_\bullet \Rightarrow \pi_{s+t} \|X_\bullet\|,$$

where $\|X_\bullet\| \in \mathcal{T}_*$ is the realization (or the diagonal, in the case of $X_\bullet \in s\mathcal{S}_*$). The spectral sequence is always defined, but X_\bullet must satisfy certain “Kan conditions” to guarantee *convergence* – see [BF, Theorem B.5].

In [DKSt2, §8.4], Dwyer, Kan and Stover showed that (2.11) coincides up to sign, from the E^2 -term on, with the spectral sequence associated to the exact couple of (2.4), which we call the *spiral spectral sequence* for X_\bullet .

If we assume that each X_n is connected, by taking loops (or applying Kan’s functor G , if $X_\bullet \in s\mathcal{S}_*$), we may replace X_\bullet by a bisimplicial group $GX_\bullet \in s\mathcal{G}$, and then (2.11) becomes the spectral sequence of $[Q]$.

3. SIMPLICIAL STEMS AND TRUNCATED SPECTRAL SEQUENCES

As noted in §1.9, the E^2 -term of any of the above equivalent spectral sequences for a simplicial space X_\bullet is determined explicitly by the simplicial 0-stem of X_\bullet .

Our goal is to extend this description to the higher terms of the spectral sequence. For this purpose, fix $n \geq 0$, and consider a simplicial Postnikov n -stem \mathcal{Q}_\bullet (which need not be realizable as $\mathcal{P}[n]X_\bullet$ for some simplicial space X_\bullet). This is equivalent to having a collection of simplicial spaces $\mathcal{Q}_\bullet^{n+k}\langle k \rangle$ for each $k \geq 0$, equipped with maps as in (1.3), with $\pi_i \mathcal{Q}_\bullet^{n+k}\langle k \rangle = 0$ for $i < k$ or $i > n+k$.

We assume that \mathcal{Q}_\bullet is *Reedy fibrant* in the sense that for each $k \geq 0$, the simplicial space $\mathcal{Q}_\bullet^{n+k}\langle k \rangle$ is Reedy fibrant. In this case, the “ n -stem version” of the spiral long exact sequence is defined as follows: for each $t, i, k \geq 0$, set $\pi_{t,i}^{(k,n)} \mathcal{Q}_\bullet := \pi_{t,i+k}^\natural \mathcal{Q}_\bullet^{n+k}\langle k \rangle$ and

$$(3.1) \quad \pi_i^{(k,n)} \mathcal{Q}_\bullet := \pi_{i+k} \mathcal{Q}_\bullet^{n+k}\langle k \rangle = \begin{cases} \pi_{i+k} \mathcal{Q}_\bullet & \text{if } 0 \leq i \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $(i+k)$ -th homotopy group $\pi_{i+k} \mathcal{Q}_\bullet$ of a Postnikov n -stem \mathcal{Q}_\bullet is well-defined, and coincides with $\pi_{i+k} X_\bullet$ for $0 \leq i \leq n$ when $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$.

3.2. Definition. The collection of long exact sequences (2.9) for $\mathcal{Q}_\bullet^{n+k}\langle k \rangle$ (indexed by $k \geq 0$):

$$(3.3) \quad \dots \Omega \pi_{t-1,*}^{\natural(k,n)} \mathcal{Q}_\bullet \xrightarrow{s_t^{(k,n)}} \pi_{t,*}^{\natural(k,n)} \mathcal{Q}_\bullet \xrightarrow{h_t^{(k,n)}} \pi_t \pi_*^{(k,n)} \mathcal{Q}_\bullet \xrightarrow{\partial_t^{(k,n)}} \Omega \pi_{t-2,*}^{\natural(k,n)} \mathcal{Q}_\bullet \dots,$$

together with the maps between adjacent k -windows induced by the map q in (1.6), will be called the *spiral n -system* of \mathcal{Q}_\bullet . When $\mathcal{Q}_\bullet = \mathcal{P}[n]X_\bullet$, we will refer to this simply as the spiral n -system of X_\bullet .

3.4. Remark. Using the exactness of (3.3), definition (3.1) implies that:

$$(3.5) \quad \pi_{t,i}^{\natural(k,n)} \mathcal{Q}_\bullet = \pi_{t,i}^{\natural(k,n)} \mathcal{Q}_\bullet^{n+k}\langle k \rangle = 0 \quad \text{for } i > n,$$

by induction on $t \geq 0$. Note, however, that while the groups $\pi_i^{(k,n)} \mathcal{Q}_\bullet$ are explicitly described by (3.1), the dependence of $\pi_{t,i}^{\natural(k,n)} \mathcal{Q}_\bullet$ on k and n requires more care.

3.6. The E^2 -term of the spectral sequence. The spiral 0-system of a simplicial Postnikov 0-stem \mathcal{Q}_\bullet reduces to a series of isomorphisms $h_t : \pi_{t,*}^{\natural(k,0)} \mathcal{Q}_\bullet \cong \pi_t \pi_*^{(k,0)} \mathcal{Q}_\bullet$ (for each $k \geq 0$). When $\mathcal{Q}_\bullet = \mathcal{P}[0]X_\bullet$ is the Postnikov 0-stem of a simplicial space X_\bullet , this allows us to identify the $E_{t,k}^2$ -term of the spiral spectral sequence for X_\bullet , which is:

$$\pi_t \pi_k X_\bullet = \pi_t \pi_k P^{0+k} X_\bullet \langle k \rangle = \pi_t \pi_k (P[0]X_\bullet)_k = \pi_t \pi_*^{(k,0)} \mathcal{P}[0]X_\bullet = \pi_t \pi_*^{(k,0)} \mathcal{Q}_\bullet,$$

with $\pi_{t,*}^{\natural(k,0)} \mathcal{Q}_\bullet = \pi_{t,*}^{\natural(k,0)} \mathcal{P}[0]X_\bullet$.

The first interesting case is the spiral 1-system, for which we have:

3.7. Proposition. *The E^3 -term of the spiral spectral sequence for a simplicial space X_\bullet is determined by the spiral 1-system of X_\bullet . In fact, $d_{t,k}^2$ may be identified with $\partial_t^{(k,1)} : \pi_t \pi_k X_\bullet \rightarrow \Omega \pi_{t-2,0}^{\natural(k,1)} X_\bullet$, while $E_{t,k}^3$ is the image of the composite map*

$$(3.8) \quad \pi_{t,0}^{\natural(k,1)} X_\bullet \xrightarrow{h_t^{(k,1)}} \pi_t \pi_k X_\bullet \cong \pi_t \pi_1^{(k-1,1)} X_\bullet \xleftarrow{\cong} \pi_{t,1}^{\natural(k-1,1)} X_\bullet \xrightarrow{s_{t+1}^{(k-1,1)}} \pi_{t+1,0}^{\natural(k-1,1)} X_\bullet.$$

Observe that (3.8) involves maps from different windows of the spiral 1-system, implicitly identified using the isomorphisms induced by the map q in (1.6).

Proof. Because $n = 1$ throughout, we abbreviate $\pi_{t,i}^{\natural(k,1)} \mathcal{Q}_\bullet$ to $\pi_{t,i}^{\natural(k)} \mathcal{Q}_\bullet$, and $\pi_i^{(k,1)} \mathcal{Q}_\bullet$ to $\pi_i^{(k)} \mathcal{Q}_\bullet$, observing that $\pi_i^{(k)} \mathcal{Q}_\bullet$ is simply $\pi_{i+k} X_\bullet$ for $i = 0, 1$, and zero otherwise, since $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$. Thus the spiral 1-system (3.3) is non-trivial for each $t \geq 1$ in (internal) degrees $i = 0, 1$ only, and we can write it in two rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_{t,1}^{\natural(k)} \mathcal{Q}_\bullet & \xrightarrow{\cong} & \pi_t \pi_1^{(k)} \mathcal{Q}_\bullet & \longrightarrow & 0 \\ & & \pi_{t-1,1}^{\natural(k)} \mathcal{Q}_\bullet & \xrightarrow{\cong} & \pi_{t-1} \pi_1^{(k)} \mathcal{Q}_\bullet & & \\ \Omega \pi_{t-1,0}^{\natural(k)} \mathcal{Q}_\bullet & \xrightarrow{s_t} & \pi_{t,0}^{\natural(k)} \mathcal{Q}_\bullet & \xrightarrow{h_t} & \pi_t \pi_0^{(k)} \mathcal{Q}_\bullet & \xrightarrow{\partial_t} & \Omega \pi_{t-2,0}^{\natural(k)} \mathcal{Q}_\bullet \\ & & \pi_{t-1,0}^{\natural(k)} \mathcal{Q}_\bullet & \xrightarrow{s_{t-1}} & \pi_{t-1} \pi_0^{(k)} \mathcal{Q}_\bullet & \xrightarrow{h_{t-1}} & \pi_{t-1} \pi_0^{(k)} \mathcal{Q}_\bullet \end{array}$$

Since $\mathcal{Q}_\bullet := \mathcal{P}[1]X_\bullet$ is the simplicial Postnikov 1-stem of X_\bullet , we actually have a collection of two-row long exact sequences, one for each k -window of $\mathcal{P}[1]X_\bullet$.

For each such k -window $\mathcal{P}_k[1]X_\bullet$, we can use the top row to identify

$$\Omega\pi_{t,0}^{\natural(k)}\mathcal{Q}_\bullet = \Omega\pi_{t,0}^{\natural(k)}\mathcal{P}_k[1]X_\bullet = \pi_{t,1}^{\natural(k)}\mathcal{P}_k[1]X_\bullet = \pi_{t,1}^{\natural(k)}\mathcal{Q}_\bullet$$

with $\pi_t\pi_1^{(k)}\mathcal{Q}_\bullet = \pi_t\pi_t^{(1)}\mathcal{P}_k[1]X_\bullet = \pi_t\pi_{k+1}X_\bullet$, so the bottom row reduces to:

$$(3.9) \quad \begin{array}{ccccccc} \pi_{t-1}\pi_{k+1}X_\bullet & \xrightarrow{s_t^{(k,1)}} & \pi_{t,0}^{\natural(k)}\mathcal{Q}_\bullet & \xrightarrow{h_t^{(k,1)}} & \pi_t\pi_kX_\bullet & \xrightarrow{\partial_t^{(k,1)}} & \pi_{t-2}\pi_{k+1}X_\bullet \\ & \searrow & \swarrow & & \swarrow & \searrow & \swarrow \\ & \text{Im}(s_t^{(k,1)}) & & \text{Im}(h_t^{(k,1)}) & & \text{Im}(\partial_t^{(k,1)}) & \\ & & & = & & & \\ & & & \text{Ker}(\partial_t^{(k,1)}) & & & \end{array}$$

Note that the following part of the E^1 -term of the exact couple for the fibration sequence $C_{n+1}P^1\Omega^iX_\bullet \rightarrow Z_nP^1\Omega^iX_\bullet$, (as in (2.4)):

$$\begin{array}{ccccccc} \pi_1 Z_{t-1}P^1\Omega^kX_\bullet & \xrightarrow{(j_{t-1})^\#} & \pi_1 C_{t-1}P^1\Omega^kX_\bullet & \xrightarrow{(d_0^{t-1})^\#} & \pi_1 Z_{t-2}P^1\Omega^kX_\bullet & \xrightarrow{(j_{t-2})^\#} & \pi_1 C_{t-2}X_\bullet \rightarrow \dots \\ \downarrow \partial_* & & & \searrow \partial_* & \downarrow & & \uparrow \text{inc} \\ & & \Omega\pi_{t-2,0}^{\natural(k)}X_\bullet = \pi_{t-2,1}^{\natural(k)}X_\bullet & & & & Z_{t-2}\pi_1P^1\Omega^kX_\bullet \\ & & \uparrow \partial_{t,0}^{(k,1)} & & \downarrow h_{t-2,1}^{(k+1,1)} \cong & & \downarrow \vartheta_{t-2} \\ & & & & & & \pi_{t-2}\pi_{k+1}X_\bullet \\ & & & & & & \\ \pi_0 Z_tP^1\Omega^kX_\bullet & \xrightarrow{(j_t)^\#} & \pi_0 C_tP^1\Omega^kX_\bullet & \xrightarrow{(d_0^t)^\#} & \pi_0 Z_{t-1}P^1\Omega^kX_\bullet & \xrightarrow{(j_{t-1})^\#} & \pi_0 C_{t-1}P^1\Omega^kX_\bullet \rightarrow \dots \\ \downarrow \hat{\vartheta}_t & \searrow & \uparrow \text{inc} & & \downarrow \hat{\vartheta}_{t-1} & \searrow & \uparrow \text{inc} \\ \pi_{t,0}^{\natural(k)}X_\bullet & & Z_t\pi_kX_\bullet & & \pi_{t-1,0}^{\natural(k)}X_\bullet & & Z_{t-1}\pi_kX_\bullet \\ & \searrow h_t^{(k,1)} & \downarrow \vartheta_t & & \searrow h_{t-1}^{(k,1)} & \downarrow \vartheta_{t-1} & \\ & & \pi_t\pi_kX_\bullet & & & & \pi_{t-1}\pi_kX_\bullet \end{array}$$

is naturally isomorphic to the exact couple for $C_{n+1}\Omega^kX_\bullet \rightarrow Z_n\Omega^kX_\bullet$, since C_{n+1} and Z_n are limits, so they commute with P^1 , and then $\pi_1P^1Z_{t-1}\Omega^kX_\bullet \cong \pi_1Z_{t-1}\Omega^kX_\bullet$, and so on. This does not imply, of course, that $\pi_{t,1}^{\natural(k)}X_\bullet \cong \pi_{t,k+1}^{\natural(k)}X_\bullet$.

We therefore see from (2.7) and (2.8) that the differential $d_{t,k}^2 : E_{t,k}^2 \rightarrow E_{t-2,k+1}^2$ may be identified with:

$$(3.10) \quad \pi_t\pi_kX_\bullet \cong \pi_t\pi_0^{(k,1)}X_\bullet \xrightarrow{\partial_{t,0}^{(k,1)}} \Omega\pi_{t-2,0}^{\natural(k)}X_\bullet = \pi_{t-2,1}^{\natural(k)}X_\bullet \xrightarrow{h_t} \pi_{t-2}\pi_1^{(k,1)}X_\bullet \cong \pi_{t-2}\pi_{k+1}X_\bullet$$

Now by definition, $E_{t,k}^3$ fits into a commutative diagram:

$$(3.11) \quad \begin{array}{ccccc} E_{t+2,k-1}^2 & \xrightarrow{d_{t+2,k-1}^2} & E_{t,k}^2 & \xrightarrow{q} & \text{Coker}(d_{t+2,k-1}^2) \\ \downarrow r & & \uparrow j & & \uparrow \kappa \\ \text{Im}(d_{t+2,k-1}^2) & \xrightarrow{\ell} & \text{Ker}(d_{t,k}^2) & \xrightarrow{s} & E_{t,k}^3 \end{array}$$

with exact rows, ℓ , j and κ monic, and thus $E_{t,k}^3 \cong \text{Im}(q \circ j)$.

From the exactness of (3.3) (together with (3.9)) we see that $\text{Coker}(d_{t+2,k-1}^2) = \text{Coker}(\partial_{t+2}^{(k-1,1)}) = \text{Im}(s_{t+1}^{(k-1,1)})$ and $\text{Ker}(d_{t,k}^2) = \text{Ker}(\partial_t^{(k,1)}) = \text{Im}(h_t^{(k,1)})$, so $E_{t,k}^3 = \text{Im}(q \circ j)$ is indeed the image of the map in (3.8). \square

3.12. Definition. An r -truncated spectral sequence is one defined up to and including the E^r -term, together with the differential $d^n : E_{t,i}^r \rightarrow E_{t-r-1,t+r}^r$, but without requiring that $d^r \circ d^r = 0$ (so the E^{r+1} -term is defined in terms of the r -truncated spectral sequence only if $d^r d^r = 0$).

The main example is the n -truncation of an (ordinary) spectral sequence (such as that of a simplicial space). In this case we do have $d^r \circ d^r = 0$, of course.

3.13. Corollary. Any Reedy fibrant simplicial Postnikov 1-stem has a well-defined 2-truncated spiral spectral sequence. Moreover, if $\mathcal{Q}_\bullet = \mathcal{P}[1]X_\bullet$ for some simplicial space X_\bullet , this 2-truncated spectral sequence coincides with the 2-truncation of the Bousfield-Friedlander spectral sequence for X_\bullet .

In general, we have a less explicit description of the higher terms in the spiral spectral sequence:

3.14. Theorem. For each $r \geq 0$, the E^{r+2} -term of the spiral spectral sequence for a simplicial space X_\bullet is determined by the spiral r -system of X_\bullet . Moreover, for any $\alpha \in E_{t,i}^{r+1}$, we have $d_{t,i}^{r+1}(\alpha) = \beta \in E_{t-r-1,i+r}^{r+1}$ if and only if α and β have representatives $\bar{a} \in \pi_t \pi_i X_\bullet$ and $\bar{b} \in \pi_{t-r-1} \pi_{i+r} X_\bullet$, respectively, such that:

$$(3.15) \quad (s_{t-2,1}^{(i,r)}) \circ (s_{t-3,2}^{(i,r)}) \circ \cdots \circ (s_{t-r,r-1}^{(i,r)}) \circ (h_{t-r-1,r}^{(i,r)})^{-1}(\bar{b}) = \partial_{t,0}^{(i,r)}(\bar{a})$$

Proof. We naturally identify $\pi_{t,k}^{(i,r)} X_\bullet$ with $\pi_{t,k+s}^{(i,r-s)} X_\bullet$ for $k \geq s$, and similarly for the maps in (3.3), so the spiral $(r-1)$ -system embeds in the spiral r -system (with an index shift).

Again we write out the E^1 -term of the spiral exact couple:

$$\begin{array}{ccccc}
\pi_r C_{t-r} P^r \Omega^i X_\bullet & \xrightarrow{(d_0^{t-r})\#} & \pi_r Z_{t-r-1} P^r \Omega^i X_\bullet & \xrightarrow{(j_{t-r-1})\#} & \pi_r C_{t-r-1} P^r \Omega^i X_\bullet \\
& & \downarrow \hat{\vartheta}_{t-r-1} & \searrow (j_{t-r-1}^{X_\bullet})\# & \uparrow \text{inc} \\
& & \Omega \pi_{t-r-1, r-1}^{\natural(i, r)} X_\bullet = \pi_{t-r-1, r}^{\natural(i, r)} X_\bullet & & Z_{t-r-1} \pi_{i+r} X_\bullet \\
& & \downarrow \ell_{t-r-1, r} & \searrow h_{t-r-1, r}^{(i, r)} \cong & \downarrow \vartheta_{t-r-1} \\
& & & & \pi_{t-r-1} \pi_{i+r} X_\bullet \\
\pi_{r-1} C_{t-r+1} P^r \Omega^i X_\bullet & \xrightarrow{(d_0^{t-r+1})\#} & \pi_{r-1} Z_{t-r} P^r \Omega^i X_\bullet & \xrightarrow{(j_{t-r})\#} & \pi_{r-1} C_{t-r} P^r \Omega^i X_\bullet \\
& & \vdots & & \vdots \\
\pi_2 C_{t-2} P^r \Omega^i X_\bullet & \xrightarrow{(d_0^{t-2})\#} & \pi_2 Z_{t-3} P^r \Omega^i X_\bullet & \xrightarrow{(j_{t-3})\#} & \pi_2 C_{t-3} P^r \Omega^i X_\bullet \\
& & \downarrow \hat{\vartheta}_{t-3} & \searrow (j_{t-3}^{X_\bullet})\# & \uparrow \text{inc} \\
& & \Omega \pi_{t-3, 1}^{\natural(i, r)} X_\bullet = \pi_{t-3, 2}^{\natural(i, r)} X_\bullet & & Z_{t-3} \pi_{i+2} X_\bullet \\
& & \downarrow \ell_{t-3, 2} & \searrow h_{t-3, 2}^{(i, r)} & \downarrow \vartheta_{t-3} \\
& & & & \pi_{t-3} \pi_{i+2} X_\bullet \\
& & \uparrow s_{t-3, 1}^{(i, r)} & & \\
\pi_1 C_{t-1} P^r \Omega^i X_\bullet & \xrightarrow{(d_0^{t-1})\#} & \pi_1 Z_{t-2} P^r \Omega^i X_\bullet & \xrightarrow{(j_{t-2})\#} & \pi_1 C_{t-2} P^r \Omega^i X_\bullet \\
& & \downarrow \hat{\vartheta}_{t-2} & \searrow (j_{t-2}^{X_\bullet})\# & \uparrow \text{inc} \\
& & \Omega \pi_{t-2, 0}^{\natural(i, r)} X_\bullet = \pi_{t-2, 1}^{\natural(i, r)} X_\bullet & & Z_{t-2} \pi_{i+1} X_\bullet \\
& & \downarrow \ell_{t-2, 1} & \searrow h_{t-2, 1}^{(i, r)} & \downarrow \vartheta_{t-2} \\
& & & & \pi_{t-2} \pi_{i+1} X_\bullet \\
& & \uparrow \partial_{t, 0}^{(i, r)} & & \\
\pi_0 Z_t P^r \Omega^i X_\bullet & \xrightarrow{(j_t)\#} & \pi_0 C_t P^r \Omega^i X_\bullet & \xrightarrow{(d_0^t)\#} & \pi_0 Z_{t-1} P^r \Omega^i X_\bullet \xrightarrow{(j_{t-1})\#} \pi_0 C_{t-1} P^r \Omega^i X_\bullet \rightarrow \dots \\
& \downarrow \hat{\vartheta}_t & \downarrow (j_t^{X_\bullet})\# & \uparrow \text{inc} & \\
& \pi_{t, 0}^{\natural(i)} X_\bullet & & Z_t \pi_i X_\bullet & \\
& \searrow h_t^{(i, r)} & & \downarrow \vartheta_t & \\
& & \pi_t \pi_i X_\bullet & &
\end{array}$$

The differential $d_{t, i}^{r+1} : E_{t, i}^{r+1} \rightarrow E_{t-r-1, i+r}^{r+1}$ may then be described as a “relation” (cf. [BK3, §3.1]) in the usual way:

Given a class $\alpha \in E_{t,i}^{r+1}$, choose a representative for it $a \in E_{t,i}^1 = \pi_0 C_t P^r \Omega^i X_\bullet$. Since it is a cycle for $d_{t,i}^1 = (j_{t-1})_\# \circ (d_0^t)_\#$, it lies in $Z_t \pi_i X_\bullet$ and thus represents an element $\bar{a} \in \pi_t \pi_i X_\bullet = E_{t,i}^2$. From the exactness of the middle row of (2.7) we see that $(d_0^t)_\#(a) \in \text{Ker}((j_{t-1})_\#) = \Omega \pi_{t-2,0}^{\natural(i,r)} X_\bullet$, and in fact $(d_0^t)_\#(a)$ represents $\partial_{t,0}^{(i,r)}(\bar{a})$. Since $\hat{\vartheta}_{t-2}$ is surjective, we can choose $e_{t-2} \in \pi_1 Z_{t-2} P^r \Omega^i X_\bullet$ mapping to $(d_0^t)_\#(a)$. Because $d_{t,i}^2(\bar{a}) = h_{t-2,1}^{(i,r)} \circ \partial_{t,0}^{(i,r)}(\bar{a})$, as in the proof of Proposition 3.7 (though $h_{t-2,1}^{(i,r)}$ need no longer be an isomorphism!), we see that it is represented by $(j_{t-2})_*(e_{t-2})$. If $r = 1$, we are done. Otherwise, we know that $d_{t,i}^2(\bar{a}) = 0$, so we can choose e_{t-2} so that $(j_{t-2})_*(e_{t-2}) = 0$, using exactness of the third column of (2.7). Again this implies that $e_{t-2} \in \text{Ker}((j_{t-2})_\#) = \Omega \pi_{t-3,1}^{\natural(i,r)} X_\bullet$, and $d_{t,i}^3(\langle a \rangle)$ is represented by $h_{t-3,2}^{(i,r)}(e_{t-2})$. Moreover, we see from (2.7) that $s_{t-3,1}^{(i,r)}(e_{t-2}) = \partial_{t,0}^{(i,r)}(\bar{a})$, using the identification $\Omega \pi_{t-2,0}^{\natural(i,r)} X_\bullet = \pi_{t-2,1}^{\natural(i,r)} X_\bullet$.

Choosing a lift to $e_{t-3} \in \pi_2 Z_{t-3} P^r \Omega^i X_\bullet$, we may assume that $(j_{t-3})_*(e_{t-3}) = 0$, so $e_{t-3} \in \Omega \pi_{t-4,2}^{\natural(i,r)} X_\bullet$ and $s_{t-4,2}^{(i,r)}(e_{t-3}) = e_{t-2}$. Continuing in this way, we finally reach $e_{t-r-1} \in \Omega \pi_{t-r-1,r-1}^{\natural(i,r)} X_\bullet$ with $s_{t-r-2,r}^{(i,r)}(e_{t-r-1}) = e_{t-r}$, and so on, and see that $d_{t,i}^{r+1}(\langle a \rangle)$ is represented by $h_{t-r-1,r}^{(i,r)}(e_{t-r-1})$. Since (as in the proof of Proposition 3.7) $h_{t-r-1,r}^{(i,r)}$ is an isomorphism, we deduce that $d_{t,i}^{r+1}(\alpha)$ is as in (3.15). \square

3.16. Remark. From the exactness of (3.3) we have $\text{Im}(\partial_{t,0}^{(i,r)}) = \text{Ker}(s_{t-1,0}^{(i,r)})$, so the image of $d_{t,i}^{r+1}$ as described in (3.15) is $\text{Ker}(\sigma_{t,i}^{r+1})$, where $\sigma_{t,i}^{r+1} := (s_{t-1,0}^{(i,r)} \circ (s_{t-2,1}^{(i,r)} \circ (s_{t-3,2}^{(i,r)} \circ \cdots \circ (s_{t-r,r-1}^{(i,r)}))$. Therefore, $E_{t+r-1,i+r}^{r+1}$ embeds naturally in $\text{Im}(\sigma_{t,i}^{r+1})$.

3.17. Corollary. *Every Reedy fibrant simplicial Postnikov r -stem has a well-defined $(r+1)$ -truncated spiral spectral sequence. If $\mathcal{Q}_\bullet = \mathcal{P}[r]X_\bullet$ for some simplicial space X_\bullet , this truncated spectral sequence coincides with the $(r+1)$ -truncation of the Bousfield-Friedlander spectral sequence for X_\bullet .*

Thus the bigraded homomorphism

$$d^{r+1} \circ d^{r+1} : E_{t,i}^r \rightarrow E_{t-2r-2,i+2r}^{r+1} \quad (t \geq 2r+2, i \geq 0)$$

serves as the first obstruction to the realizability of the simplicial Postnikov r -stem \mathcal{Q}_\bullet by a simplicial space X_\bullet .

4. A COSIMPLICIAL VERSION

There are actually four variants of the above spectral sequence which we might consider, for a simplicial or cosimplicial object over simplicial or cosimplicial sets. The case of bicosimplicial sets is in principle strictly dual to that of bisimplicial sets, but because the category of cosimplicial *sets* has no (known) useful model category structure, we must restrict to bicosimplicial abelian groups – or equivalently, ordinary double complexes. Thus the main new case of interest is that of cosimplicial simplicial sets, or *cosimplicial spaces*.

4.1. The spectral sequence of a cosimplicial space. If $X^\bullet \in c\mathcal{S}_*$ is a fibrant cosimplicial pointed space with total space $\text{Tot } X^\bullet$, there are various constructions for the homotopy spectral sequence of X^\bullet :

- (a) Using the tower of fibrations for $(\text{Tot}_n X^\bullet)_{n=0}^\infty$ (cf. [BK1, X, §6]).
- (b) Using “relations” on the normalized cochains $N^n \pi_t X^\bullet := \pi_t X^n \cap \text{Ker}(s^0) \cap \dots \cap \text{Ker}(s^{n-1})$ (cf. [BK3, §7]).
- (c) Using a cofibration sequence dualizing (2.4) (cf. [R, §3]).

Bousfield and Kan showed that the result is essentially unique (see [BK3]). Since the main ingredient needed for to define the spiral exact couple is the diagram (2.7), we use the first approach:

4.2. Definition. For any Reedy fibrant cosimplicial pointed space $X^\bullet \in c\mathcal{S}_*$, consider the fibration sequence

$$(4.3) \quad F_n X^\bullet \xrightarrow{j_n} \text{Tot}_n X^\bullet \xrightarrow{p_n} \text{Tot}_{n-1} X^\bullet,$$

where $\text{Tot}_n X^\bullet := \text{map}_{c\mathcal{S}_*}(\text{sk}_n \Delta, X^\bullet)$ and the fibration p_n is induced by the inclusion of cosimplicial spaces $\text{sk}_{n-1} \Delta \hookrightarrow \text{sk}_n \Delta$.

The cokernel of $(j_n)_\# : \pi_* F_n X^\bullet \hookrightarrow \pi_* \text{Tot}_n X^\bullet$ is called the n -th *natural (graded) cohomotopy group* of X^\bullet , and denoted by $\pi_{\natural*}^n X^\bullet$.

4.4. Remark. We may identify $F_n X^\bullet$ with the looped normalized cochain object $\Omega^n N^n X^\bullet$, where

$$(4.5) \quad N^n X^\bullet := X^n \cap \text{Ker}(s^0) \cap \dots \cap \text{Ker}(s^{n-1}),$$

and $\pi_* N^n X^\bullet$ with $N^n \pi_* X^\bullet$ (see [BK1, X, Proposition 6.3]).

Moreover, the composite

$$\pi_{*+1} \Omega^n N^n X^\bullet \cong \pi_{*+1} F_n X^\bullet \xrightarrow{(j_n)_\#} \pi_{*+1} \text{Tot}_n X^\bullet \xrightarrow{\partial_n} \pi_* F_{n+1} X^\bullet \cong \pi_* \Omega^{n+1} N^{n+1} X^\bullet$$

(where ∂_n is the connecting homomorphism for the (4.3)), may then be identified with the differential

$$(4.6) \quad \delta^n := \sum_{i=0}^n (-1)^i d^i : N^n \pi_* X^\bullet \rightarrow N^{n+1} \pi_* X^\bullet,$$

for the normalized cochain complex $N^* \pi_* X^\bullet$, so that

$$(4.7) \quad \text{Ker}(\delta^n) / \text{Coker}(\delta^{n+1}) \cong \pi^n \pi_* X^\bullet$$

(cf. [BK1, X, §7.2]).

4.8. Proposition. *For any pointed cosimplicial space X^\bullet there is a natural spiral long exact sequence:*

$$(4.9) \quad \begin{aligned} \dots \rightarrow \Omega \pi_{\natural*}^{n-1} X^\bullet &\xrightarrow{s^n} \pi_{\natural*}^n X^\bullet \xrightarrow{h^n} \pi^n \pi_* X^\bullet \xrightarrow{\partial^n} \Omega \pi_{\natural*}^{n-2} X^\bullet \\ &\xrightarrow{s^{n-1}} \pi_{\natural*}^{n-1} X^\bullet \rightarrow \dots \rightarrow \pi_{\natural*}^0 X^\bullet \xrightarrow{\cong} \pi^0 \pi_* X^\bullet \end{aligned}$$

Proof. By choosing a fibrant replacement in the model category of cosimplicial simplicial sets defined in [BK1, X, §5], if necessary, we may assume that X^\bullet is Reedy fibrant. We then obtain a commutative diagram as in (2.7) with exact rows and columns:

$$(4.10) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}(j_n)_* & \hookrightarrow & B^{n+1}X^\bullet & \xrightarrow{(j_n)_*} & B^{n+1}\pi_*X^\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Omega\pi_{\natural}^{n-1}X^\bullet & \xrightarrow{\ell_{n-1}} & \pi_*\text{Tot}_nX^\bullet & \xrightarrow{(j_n^{X^\bullet})_\#} & Z^n\pi_*X^\bullet \longrightarrow \text{Coker } h^n \longrightarrow 0 \\ & & \downarrow & \searrow s_n & \downarrow \hat{\vartheta}_n & & \downarrow \vartheta_n \\ 0 & \longrightarrow & \text{Ker } h^n & \hookrightarrow & \pi_{\natural}^nX^\bullet & \xrightarrow{h_n} & \pi^n\pi_*X^\bullet \longrightarrow \text{Coker } h^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

in which $B^{n+1}X^\bullet := \text{Im}(j_{n+1})_\# \subseteq \pi_*\text{Tot}_nX^\bullet$ and $B^{n+1}\pi_*X^\bullet := \text{Im}(\delta^{n+1}) = \text{Im}(\partial_{n+1} \circ (j_{n+1})_\#)$ are the respective coboundary objects.

The construction of the maps h^n , s^n , and ∂^n , and the proof of the exactness of (4.9), are then precisely as in §2.6. \square

4.11. Definition. The *spiral n -system* of a pointed cosimplicial space $X^\bullet \in c\mathcal{S}_*$ is defined to be the collection of long exact sequences (4.9) for the Postnikov n -stem functor $\mathcal{P}[n]$ applied to X^\bullet , one for each k -window of $\mathcal{P}[n]X^\bullet$.

As in Definition 3.2, this may actually be defined for a cosimplicial Postnikov n -stem \mathcal{P}^\bullet , not necessarily realizable as $\mathcal{P}^\bullet = \mathcal{P}[n]X^\bullet$.

By construction, the homotopy spectral sequence of a (fibrant) cosimplicial space X^\bullet , obtained as in (4.1), is associated to the spiral exact couple (4.9). The proofs of Proposition 3.7 and Theorem 3.14 use only the description of the spiral exact couple for X_\bullet derived from (4.10), so by using (4.10) instead we can prove their analogues in the cosimplicial case, and show:

4.12. Theorem. *The E_{r+2} -term of the homotopy spectral sequence for a cosimplicial space X^\bullet is determined by the spiral r -system of X^\bullet .*

An analogue of Corollary 3.17 also holds, as well as:

4.13. Proposition. *The differential $d_2^{t,i} : E_2^{t,i} \rightarrow E_2^{t+2,i+1}$ may be identified with $\partial_{(i,1)}^t : \pi^t\pi_iX^\bullet \rightarrow \Omega\pi_{\natural(i)}^{t+2,0}X^\bullet$.*

4.14. Examples. As noted in the introduction, many commonly used spectral sequences arise as the spiral spectral sequence of an appropriate (co)simplicial space, so Theorems 3.14 and 4.12 allow us to extract their E^r - or E_r -terms from the appropriate spiral systems. For instance:

- (a) Segal's homology spectral sequence (cf. [Se]), the van Kampen spectral sequence (cf. [St]), and the Hurewicz spectral sequence (cf. [Bl1]) are constructed using bisimplicial sets.
- (b) The unstable Adams spectral sequences of [BCKQRS, BK2] and [BCM, §4], Rector's version of the Eilenberg-Moore spectral sequence (cf. [R]), and Anderson's generalization of the latter (cf. [An]) are all associated to cosimplicial spaces.
- (c) The usual construction of the stable Adams spectral sequence for $\pi_*^s X \otimes \mathbb{Z}/p$ (cf. [Ad, §3]) uses a tower of (co)fibrations, rather than a cosimplicial space, but when X is finite dimensional, it agrees in a range with the unstable version for $\Sigma^N X$, so Theorem 4.12 applies stably, too.

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